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New techniques for solving some matrix and matrix differential equations

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Abstract Matrix and matrix differential equations play an important role in system theory, control theory, stability theory of differential equations, communication systems and many other fields. In this paper, we present the solutions of non-homogeneous matrix differential equations, convolution matrix differential equations and matrix equations which include the renewal matrix equation by using convolution and Kronecker products of matrices. Furthermore, the existence and uniqueness of the solution of some important and interesting special cases of these equations are also considered with some illustrated examples in order to show our new approaches.

1. Introduction and preliminary results

In addition to the matrix usual multiplication; there has been renewed interest in two kinds of matrix multiplication. These multiplications are the convolution and Kronecker products which are playing very important roles in many applications and the technique has successfully applied in various fields of pure and applied mathematics, for example, in the solution of matrix and matrix differential equations [1–15]. The notations: $M_{m,n}$, A^T, A^{-1}, A^+, $rank(A)$, e^A, $||A||$, $\sigma(A)$ are stand to the set of all $m \times n$ matrices (when $m = n$, we write M_n instead of $M_{n,n}$), transpose, inverse, Moore–Penrose inverse, rank, exponential, norm, and the set of all eigenvalues of a matrix A, respectively. Now, we recall the main definitions and some important properties of the Kronecker and convolution products of matrices that will be very useful in our investigation in the solution of matrix equations and matrix differential equations.

The Kronecker and convolution products used in many fields are almost as important as the usual product. One of the principle reasons is that these products affirming their capability of solving a wide range of problems and playing important tools in many fields such as control theory, system theory, statistics, physics, communication systems, optimization, economics and engineering. These include signal processing, image processing, semi definite programming, matrix equations, matrix differential equations and many other applications [1–21]. The following four matrix operations are studied by many researchers [1–4,7–14,20,21] and defined as follow:

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(i) *Kronecker product:*

$$A \otimes B = (a_{ij}B)_{ij} \in M_{mp,nq}, \tag{1-1}$$

where $A = (a_{ij}) \in M_{m,n}$ and $B = (b_{kl}) \in M_{p,q}$.

(ii) *Kronecker sum:*

$$A \oplus B = (A \otimes I_n) + (I_m \otimes B) \in M_{mn}, \tag{1-2}$$

where $A = (a_{ij}) \in M_m$ and $B = (b_{kl}) \in M_n$.

(iii) *Vector operator:*

$$VecA = (a_{11}a_{21}\ldots a_{m1}a_{12}a_{22}\ldots a_{m2}\ldots a_{1n}a_{2n}\ldots a_{mn})^T \in M_{mn,1}, \tag{1-3}$$

where $A = (a_{ij}) \in M_{m,n}$.

(iv) *Convolution product:*

$$A * B(t) = (h_{ir}(t)) \text{ with } h_{ir}(t) = \sum_{k=1}^{n}\int_0^t f_{ik}(t-x)g_{kr}(x)dx = \sum_{k=1}^{n} f_{ik} * g_{kr}(t), \tag{1-4}$$

where $A(t) = (f_{ij}(t)) \in M_{m,n}$ and $B(t) = (g_{jr}(t)) \in M_{n,p}$ are integrable matrices for all $t \geqslant 0$, such that $f_{ij}(t)$ and $g_{jr}(t)$ are well-defined functions for all positive integer values i, j, r.

The following three definitions are also very useful in our investigation in the solutions of renewal matrix equations and matrix differential convolution equations. If $A(t) = (f_{ij}(t)) \in M_n$ is an integrable matrix, then [5–6,20]

(i) The *m-power matrix convolution product* of $A(t)$ is defined by

$$A^{\{m\}}(t) = A * A * \cdots * A(t) = (f_{ij}^{\{m\}}(t)) \in M_n \text{ with } f_{ij}^{\{m\}}(t) = \sum_{k=1}^{n} f_{ik}^{\{m-1\}} * f_{kj}(t), \tag{1-5}$$

where m is positive integer number, and $A^{\{1\}}(t) = A(t)$.

(ii) The determinant of $A(t)$ is defined by

$$\det A(t) = \sum_{j=1}^{n}(-1)^{j+1} f_{1j} * D_{1j}, \tag{1-6}$$

where D_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix function obtained from $A(t)$ by deleting row i and column j of $A(t)$. We call D_{ij} the minor of $A(t)$ corresponding to the entry $f_{ij}(t)$ of $A(t)$.

(iii) If $\det(A(t)) \neq 0$, the inversion of $A(t)$ is defined by

$$A^{\{-1\}}(t) = (h_{ij}(t)) \text{ with } h_{ij}(t) = [\det(A(t))]^{\{-1\}} * adjA(t). \tag{1-7}$$

For any compatibly matrices A, B, C and D, we shall make frequent use of the following properties of the Kronecker products [1–4,7–14].

(i) $(A \otimes B)^T = A^T \otimes B^T$ (1-8)

(ii) $(A \otimes B)^+ = A^+ \otimes B^+$ (1-9)

(ii) $rank(A \otimes B) = rank(A)rank(B)$ (1-10)

(iii) $e^{(A\oplus B)} = e^A \otimes e^B$ (1-11)

(iv) $(A \otimes B)(C \otimes D) = AC \otimes BD$ (1-12)

(v) $I_m \otimes I_n = I_n \otimes I_m = I_{mn}$, where I_m is the identity *matrix of order* $m \times m$. (1-13)

(vi) If $\sigma(A) = \{\lambda_i : i = 1, 2, \ldots, m\}$ and $\sigma(B) = \{\mu_j : j = 1, 2, \ldots, n\}$ are the set of eigenvalues of $A \in M_m$ and $B \in M_n$, respectively. Then

(i) $\sigma(A \otimes B) = \{\lambda_i\mu_j : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$. (1-14)

(ii) $\sigma(A \oplus B) = \{\lambda_i + \mu_j : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$. (1-15)

(vii) $f(A \otimes I_n) = f(A) \otimes I_n, f(I_n \otimes A) = I_n \otimes f(A)$, (1-16)

where f is analytic function on the region containing the eigenvalues of $A \in M_m$ such that $f(A)$ exist.

Some special cases include (1-16):

(i) $e^{A\otimes I} = e^A \otimes I$ and $e^{I\otimes A} = I \otimes e^A$ (1-17)

(ii) $\sinh(A \otimes I) = \sinh(A) \otimes I$ and $\sinh(I \otimes A) = I \otimes \sinh(A)$ (1-18)

(iii) $\cosh(A \otimes I) = \cosh(A) \otimes I$ and $\cosh(I \otimes A) = I \otimes \cosh(A)$. (1-19)

For any matrix $A \in M_m$, the spectral representation of e^A and e^{At} assures that:

$$e^A = \sum_{i=0}^{n} x_i y_i^T e^{\lambda_i}, e^{At} = \sum_{i=0}^{n} x_i y_i^T e^{\lambda_i t}, \tag{1-20}$$

where $\{\lambda_1, \cdots, \lambda_n\}$ are the eigenvalues of A, $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are the set of all eigenvectors of A and A^T, respectively, corresponding to the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.

The nice relationship between the Kronecker product and vector-operator is given by [1–4,7–14]

$$Vec(AXB) = (B^T \otimes A)VecX, \tag{1-21}$$

where $A \in M_{m,n}$, $B \in M_{p,q}$ and $X \in M_{n,p}$.

For any compatibly integrable matrices $A(t) = (f_{ij}(t))$ and $B(t) = (g_{ij}(t))$, we shall make frequent use of the following properties of the convolution product [18,20,21]:

(i) $\pounds(A * B) = \pounds((A))(s)\pounds((B))(s)$ (1-22)

(ii) $(A * B(t))(i, r) \leqslant (A(t)B(t))(i, r)$ (t fixed) (1-23)

(iii) $\|A * B(t)\| \leqslant \|A(t)\|.\|B(t)\|$ (1-24)

(iv) $\|A^{\{m\}}(t)\| \leqslant \|A(t)\|^m$ (m is positive integer) (1-25)

Finally, the Moore–Penrose inverse is widely used in perturbation theory, singular systems, neural network problems, least-squares problems, optimization problems and many other subjects. The Moore–Penrose inverse of an arbitrary matrix $A \in M_{m,n}$ is defined to be the unique solution of the following four matrix equations [4,11,14]:

$$AXA = A, \; XAX = X, \; (AX)^T = AX, \; (XA)^T = XA, \tag{1-26}$$

and is often denoted by $X = A^+ \in M_{n,m}$. Note that if $A \in M_{m,n}$, then we have the following special cases:

(i) If $rank(A) = m = n$, then $A^+ = A^{-1}$ (1-27)

(ii) If $rank(A) = n$, $A^+ = (A^TA)^{-1}A^T$ and $A^+A = I_n$ (1-28)

(iii) If $rank(A) = m$, $A^+ = A^T(AA^T)^{-1}$ and $AA^+ = I_m$. (1-29)

In the present paper, based on the vector-operator, Kronecker products and convolution products of matrices, we present the general solutions of some matrix and matrix differential equations. These equations involve the renewal matrix equation, general matrix equation and non-homogeneous matrix differential equations, then we show that the solutions of some of these equations can be written in convolution or Kronecker forms. Finally, the existence and uniqueness of the solution of some important and interesting special cases with some illustrated examples are discussed and given.

2. Renewal matrix equation

In this section, we present the solutions of renewal matrix by using the convolution product of matrices.

We will use our knowledge of the Markov renewal process and semi-Markov kernel as follows [16–21]: Let $E = \{1, 2, \ldots, N\}$ be a set representing the states of a system and probability space with probability function P on which we define a bivariate time-homogeneous Markov chain $(X, T) = \{(X_n, T_n)\}_{n=1}^{\infty}$, X_n takes values on E and T_n on the half-real line $R_+ = [0, \infty)$ with $0 = T_0 \leqslant T_1 \leqslant \cdots \leqslant T_n \leqslant T_{n+1} \leqslant \cdots$. Now the chain, $U_n = T_n - T_n - 1, \forall\ n \geqslant 1$ is called a Markov renewal process with transition function and the semi-Markov kernel:

$$Q_{ij}(t) = P\{X_{n+1} = j, U_n \leqslant t, X_n = i\}, \tag{2-1}$$

where $i, j \in E, i \neq j, t \geqslant 0$ and $Q_{ii}(t) = 0\ (i \in E, t \geqslant 0)$.

The first component X is representing to the Markov chain with the transition function $P(i,j) = Q_{ij}(\infty)$ and $T_n = U_1 + U_2 + \cdots + U_n$ is the time of the n-th renewal with $N_t = \sup\{n : T_n \leqslant t\}$ a counter process of the number of renewal in [0, t]. We define the semi-Markov process $Z = \{Z_t : t \in R_+\}$, associated with the MRP defined above, by $Z_t = X_{N_t}$ (and $X_n = Z_{T_n}$).

Define now the transition probability that system occupied state $j \in E$ at time $t \geqslant 0$ given that it started at state i at time $t = 0$, $P_{ij}(t) = P\{Z_t = j, Z_0 = i\} = P\{X_{N_t} = j, X_0 = i\}$.

Now, the transition probabilities verify the following renewal matrix equation:

$$P_{ij}(t) = h_i(t)1_{\{i=j\}} + \sum_k \int_0^t Q_{ik}(x)P_{kj}(t-x)dx, \tag{2-2}$$

where $h_i(t) = 1 - \sum_k Q_{ik}, t \geqslant 0$.

Now, we can rewrite the renewal Eq. (2-2) in matrix convolution form as follows:

$$P(t) = h(t) + P * Q(t), \tag{2-2}$$

where $P(t) = [P_{ij}(t)]$ is a square matrix function, $h(t) = diag(h_i(t))$ is a diagonal matrix function and $Q(t)$ is the semi-Markov matrix.

Theorem 2.1. *If $||.||$ is matrix norm and $||Q|| < 1$. Then the solution of renewal matrix equation in (2-2) is given by:*

$$P(t) = h(t) * (I - Q(t))^{\{-1\}}, \tag{2-3}$$

where $I = I(t) = diag(1(t))$ diagonal matrix, $1(t)$ is the step function.

Proof. By applying the Laplace transforms of renewal matrix equation in (2-2), we have

$$£\{P(t)\} = £\{h(t)\} + £\{P(t) * Q(t)\}. \tag{2-4}$$

Set $Y(s) = £\{P(t)\}$, $H(s) = £\{h(t)\}$ and $C(s) = £\{Q(t)\}$, then the Eq. (2-4) can be written as:

$$Y(s) = H(s) + Y(s)C(s). \tag{2-5}$$

Hence,

$$Y(s) = H(s)(I - C(s))^{\{-1\}}. \tag{2-6}$$

By taking $£^{-1}$ of both sides in Eq. (2-6) gives the result in (2-3) and the proof of Theorem 2.1 is completed. □

Note that the matrix $(I - Q(t))^{\{-1\}}$ can be obtained by two ways, either by truncated series development or by explicit inversion within the above described convolution algebra. In the first case, we have

$$(I - Q(t))^{\{-1\}} = I + \sum_{k=1}^{n} Q^{\{k\}}(t) + R_n(t), \tag{2-7}$$

where $Q_{ij}^{\{n+1\}}(t) = \sum_{k \in E} \int_0^t Q_{ik}(u) \cdot Q_{kj}^{\{n\}}(t-u)du,\ n = 1,2,\ldots,$ $Q^{\{1\}}(t) = Q(t)$,

$$\begin{aligned} R_n(t) &= \sum_{k=n+1}^{\infty} Q^{\{k\}}(t) \\ &= Q^{\{n+1\}}(t) * (I + Q(t)) + Q^{\{2\}} + \cdots \\ &= Q^{\{n+1\}}(t) * (I - Q(t))^{\{-1\}}, \end{aligned} \tag{2-8}$$

where $R_n(t)$ is the total error of $P(t)$.

By using Lemma 1.3, we have for any matrix norm $||.||$

$$\begin{aligned} \|Q^{\{n\}}(t)\| = \|Q^{\{n-1\}}(t).Q(t)\| &\leqslant \|Q^{\{n-1\}}(t)\|.\|Q(t)\| \leqslant \cdots \\ &\leqslant \|Q(t)\|^n. \end{aligned} \tag{2-9}$$

Thus the total error on $P(t)$ due to the above truncation is also bounded by (2-9), and if $||Q(t)|| < 1$, then $\|R_n(t)\| \leqslant \frac{\|Q(t)\|^{n+1}}{1 - \|Q(t)\|}$. Thus for given maximum error, ε say, we have

$$n + 1 \geq \frac{\ln\{\varepsilon(1 - \|Q(t)\|)\}}{\ln \|Q(t)\|}. \tag{2-10}$$

In the direct inversion method, we shall calculate $(I - Q(t))^{\{-1\}}$ directly:

$$(I - Q(t))^{\{-1\}} = (\det(I - Q(t)))^{\{-1\}} * adj(I - Q(t)). \tag{2-11}$$

Example 2.2. Alternating renewal process as a Markov renewal process. Consider an alternating renewal process: (X_n, Y_n), $n = 1, 2, \cdots$, with distribution functions F and G on $R_\infty = [0, \infty)$. Then the semi Markov matrix is: $Q = \begin{bmatrix} 0 & F \\ G & 0 \end{bmatrix}$, and consequently,

$$(I - Q(t))^{\{-1\}} = (1 - F * G)^{\{-1\}} * \begin{bmatrix} 1 & F \\ G & 1 \end{bmatrix},$$

$$I - diag(Qv) = \begin{bmatrix} 1 - F & 0 \\ 0 & 1 - G \end{bmatrix}$$

where $v = [1, 1, \cdots, 1]^T$ is the unit vector with appropriate dimension. The renewal function L for $t \geqslant 0$ is given by:

$$\begin{aligned} L(t) &= 1(t) + H(t) + H^{\{2\}}(t) + \cdots + H^{\{n\}}(t) + \cdots \\ &= (1 - F * G)^{\{-1\}} \quad (H = F * G). \end{aligned}$$

The transition function is given by:

$$P(t) = L * \begin{bmatrix} 1 & F \\ G & 1 \end{bmatrix} * \begin{bmatrix} 1 - F & 0 \\ 0 & 1 - G \end{bmatrix}$$

$$= L * \begin{bmatrix} 1 - F & G * (1 - G) \\ G * (1 - F) & 1 - G \end{bmatrix},$$

or $P_{11}(t) = L * (1 - F)(t)$, $P_{12}(t) = L * F * (1 - G)(t)$, $P_{21}(t) = L * G * (1 - F)(t)$, $P_{22}(t) = L * (1 - G)(t)$.

These results represent the availability formula depending on the initial conditions in the standard alternating renewal process.

3. Some matrix differential equations

In this section, we present the solutions of some important matrix differential equations by using the Kronecker and convolution products of matrices.

The simplest homogeneous linear matrix differential equation:

$$X'(t) = AX(t),\ X(t_0) = C, \tag{3-1}$$

where $A \in M_n$ and $C \in M_{n,m}$ are given scalar matrices, and $X(t) \in M_{n,m}$ is the unknown matrix to be solved. In fact, the general solution of (3-1) is given by [1–4]:

$$X_c(t) = e^{A(t-t_0)}C. \tag{3-2}$$

Now we show that the solutions of the following matrix differential equations can be written in convolution forms as in next results.

Theorem 3.1. *The general solution of the following non-homogeneous matrix differential equation:*

$$X'(t) = AX(t) + U(t),\ X(t_0) = C, \tag{3-3}$$

is given by:

$$\begin{aligned} X(t) &= e^{A(t-t_o)}C + \int_{t_0}^{t} e^{A(t-s)}U(s)ds \\ &= e^{A(t-t_0)}C + e^{At} * U(t), \end{aligned} \tag{3-4}$$

where $A \in M_n$ and $C \in M_{n,m}$ are given scalar matrices, $U(t) \in M_{n,m}$ is a given matrix function and $X(t) \in M_{n,m}$ is the unknown matrices to be solved.

Proof. Suppose that $X_p(t) = e^{At}G(t)$ is the particular solution of the Eq. (3-3). Then the derivative of $X_p(t)$ is given by:

$$X_p'(t) = e^{At}G'(t) + Ae^{At}G(t). \tag{3-5}$$

Substituting (3-5) in (3-3), we obtain

$$e^{At}G'(t) + Ae^{At}G(t) = Ae^{At}G(t) + U(t). \tag{3-6}$$

Thus,

$$G'(t) = e^{-At}U(t). \tag{3-7}$$

Integrating both sides of (3-7) between t_0 and t gives:

$$G(t) = \int_{t_0}^{t} e^{-As}U(s).ds \tag{3-8}$$

Hence, by assumption, we conclude that the particular solution of Eq. (3-3) is

$$\begin{aligned} X_p(t) &= e^{At}G(t) = e^{At}\int_{t_0}^{t} e^{-As}U(s).ds \\ &= \int_{t_0}^{t} e^{A((t-s)}U(s)ds = e^{At} * U(t). \end{aligned} \tag{3-9}$$

Hence, from (3-2) and (3-9) we get the solution as in (3-4). □

Theorem 3.2. *Consider the following matrix differential equation:*

$$X'(t) = AX(t) + X(t)B + U(t),\ X(t_0) = C, \tag{3-10}$$

where $A \in M_n$, $B \in M_m$ and $C \in M_{n,m}$ are given scalar matrices, $U(t) \in M_{n,m}$ is a given matrix function and $X(t) \in M_{n,m}$ is unknown matrix to be solved. Then the general solution of (3-10) is given by:

$$X(t) = e^{At}Ce^{Bt} + (e^{At}U(t)) * e^{Bt}. \tag{3-11}$$

Proof. If we use the $Vec(.)$ notation of (3-10), then by using (1-21), we get the following equivalent equation:

$$\begin{aligned} Vec(X'(t)) &= (I_m \otimes A + B^T \otimes I_n)VecX(t) + VecU(t) \\ &= (B^T \oplus A)VecX(t) + VecU(t). \end{aligned} \tag{3-12}$$

Note that the Eq. (3-12) can be rewritten as follows:

$$x'(t) = Hx(t) + u(t),\ x(0) = c, \tag{3-13}$$

where $x'(t) = VecX'(t)$, $H = B^T \oplus A$, $x(t) = VecX(t)$, $c = VecC$ and $u(t) = VecU(t)$.

Now by Theorem 3.1, we have

$$VecX(t) = x(t) = e^{Ht}c + e^{Ht} * u(t) = e^{(B^T \oplus A)t}c + e^{(B^T \oplus A)t} * u(t)$$

$$= (e^{B^T t} \otimes e^{At})VecC + (e^{B^T t} \otimes e^{At}) * VecU(t)$$

$$= Vec(e^{At}Ce^{Bt}) + Vec((e^{At}U(t)) * e^{Bt})$$

$$= Vec(e^{At}Ce^{Bt} + e^{At} * U(t) * e^{B(t)})$$

Hence, $X(t) = e^{At}Ce^{B} + (e^{At}U(t)) * e^{Bt}$. □

Theorem 3.3. *Let $A \in M_n$ be a given scalar positive semi definite matrix, $C \in M_{n,m}$ and $D \in M_{n,m}$ be given scalar matrices, $U(t) \in M_{n,m}$ be a given matrix function, and $X(t) \in M_{n,m}$ be the unknown matrix to be solved. Then*

(i) The general solution of the following matrix convolution differential equation:

$$X'(t) = A * X(t) + U(t),\ X(0) = C \tag{3-14}$$

is given by:

$$X(t) = \cosh(Bt) * U(t) + (\cosh Bt)C. \tag{3-15}$$

(ii) The general vector solution of the following matrix convolution differential equation:

$$X'(t) = A * X(t) * B + U(t),\ X(0) = C \tag{3-16}$$

is given by:

$$\begin{aligned} VecX(t) &= (\cosh((B^T)^{1/2} \otimes A^{1/2})t) * VecU(t) \\ &\quad + (\cosh((B^T)^{1/2} \otimes A^{1/2})t)VecC. \end{aligned} \tag{3-17}$$

Proof. (i) Follows immediately by applying Laplace transforms.

(ii) By using the $Vec(.)$-notation of (3-16), then by using (1-21), we get the following equivalent equation:

$$\begin{aligned} Vec(X'(t)) &= Vec[A * X(t) * B + U(t)] \\ &= (B^T \otimes A) * VecX(t) + VecU(t). \end{aligned} \tag{3-18}$$

Note that the Eq. (3-18) can be rewritten as follows:

$$x'(t) = H * x(t) + u(t),\ x(0) = c, \tag{3-19}$$

where $x'(t) = VecX'(t)$, $H = B^T \otimes A$, $x(t) = VecX(t)$, $u(t) = VecU(t)$ and $c = Vec(C)$.

Now by part (i) of Theorem 3.3, then the solution of (3-19) is given by:

$$x(t) = (\cosh H^{1/2}t) * u(t) + (\cosh H^{1/2}t)c. \tag{3-20}$$

Now the Eq. (3-20) gives the vector solution as in (3-17). $\square$

Corollary 3.4. *Let $A, B, C \in M_n$ be given scalar positive definite matrices, $U(t) \in M_n$ be a given matrix function, and $X(t) \in M_n$ be the unknown matrix function to be solved. Then*

(i) The general solution of the following matrix convolution differential equation:

$$X'(t) = I * X(t) * B,\ X(0) = C \tag{3-21}$$

is given by:

$$X(t) = C(\cosh{(B^{1/2})}^T t)^T. \tag{3-22}$$

(ii) The general solution of the following matrix convolution differential equation:

$$X'(t) = A * X(t) * I,\ X(0) = C \tag{3-23}$$

is given by:

$$X(t) = (\cosh A^{1/2}t)C. \tag{3-24}$$

Proof. (i) Set $A = I$ and $U(t) = 0$ in (3-16) and (3-17), we have

$$\begin{aligned} VecX(t) &= (\cosh((B^T)^{1/2} \otimes I)t)VecC \\ &= \{(\cosh{(B^T)}^{1/2}t) \otimes I\}VecC \\ &= Vec\{IC(\cosh{(B^T)}^{1/2}t)^T\}. \end{aligned}$$

Hence, $X(t) = C(\cosh{(B^{1/2})}^T t)^T$.

(ii) Set $B = I$ and $U(t) = 0$ in (3-16) and (3-17), we have

$$\begin{aligned} VecX(t) &= (\cosh(I \otimes A)t)VecC \\ &= (I \otimes \cosh A^{1/2}t)VecC \\ &= Vec\{(\cosh A^{1/2}t)CI\} \end{aligned}$$

Hence, $X(t) = (\cosh A^{1/2}t)C$. $\square$

4. Some matrix equations

The general linear matrix equation is given by:

$$\sum_{i=1}^{p} A_i X B_i = C, \tag{4-1}$$

If we use the Vec-notation of (4-1), then by using (1-21) we have the following equivalent equation:

$$\sum_{i=1}^{p} (B_i^T \otimes A_i) VecX = VecC. \tag{4-2}$$

Note that the unique solution of (4-2) is obtained if and only if $\sum_{i=1}^{p}(B_i^T \otimes A_i)$ is invertible, but if $\sum_{i=1}^{p}(B_i^T \otimes A_i)$ is not-invertible, then we consider the rank of the following augmented matrix:

$$\left[\sum_{i=1}^{n} (B_i^T \otimes A_i) : VecC\right]. \tag{4-3}$$

Now, if $rank[\sum_{i=1}^{n}(B_i^T \otimes A_i) : VecC] = rank[\sum_{i=1}^{p}(B_i^T \otimes A_i)]$, then the matrix equation as in (4-2) has a solution; otherwise there is no solution.

Definition 4.1. *If $A \in M_{m,n}$ and $B \in M_{p,q}$ $(m \geqslant n, p \geqslant q)$, then the least-square problem of the Kronecker equation: $(A \otimes B)x = c$ is defined as the following form:*

$$\min_x \|c - (A \otimes B)\|_2^2 \equiv \min_X \|C - BXA^T\|_2^2, \tag{4-4}$$

where $x = VecX \in M_{nq,1}$, $c = VecC \in M_{mp,1}$ and $\|A\|_2 = \sqrt{\sum_{i,j=1}^{n,m} |a_{ij}|^2} = \{tr(A^T A\}^{1/2}$.

Note that the relationship between $X \in M_{q,n}$ and $x \in M_{nq,1}$ in Definition 4.1 is given by:

$$X = [x^{(1)}, x^{(2)}, \ldots, x^{(n)}] = \begin{bmatrix} x_1 & x_{q+1} & \cdots & x_{(n-1)q+1} \\ . & . & & . \\ . & . & & . \\ x_q & x_{2q} & \cdots & x_{nq} \end{bmatrix}_{q \times n}. \tag{4-5}$$

Also the relationship between $C \in M_{p,m}$ and $c \in M_{mp,1}$ in Definition 4.1 is given by:

$$C = [c^{(1)}, c^{(2)}, \ldots, c^{(m)}] = \begin{bmatrix} x_1 & x_{p+1} & \cdots & x_{(m-1)p+1} \\ . & . & & . \\ . & . & & . \\ x_p & x_{2p} & \cdots & x_{mp} \end{bmatrix}_{p \times m}. \tag{4-6}$$

Theorem 4.2. *Let $A \in M_{m,n}$ $(m \geqslant n)$ with $rank(A) = n$ and $B \in M_{p,q}$ $(p \geqslant q)$ with $rank(B) = q$. Then the solution of the least-square problem as in (4-4) is given by:*

$$X = B^+ C(A^+)^T, \tag{4-7}$$

where $A^+ = (A^T A)^{-1} A^T$ and $B^+ = (B^T B)^{-1} B^T$.

Proof. Since $rank(A) = n$ and $rank(B) = q$, then $rank(A \otimes B) = nq$ and the normal equation which is related to the least-square problem is given by:

$$(A \otimes B)^T (A \otimes B)x = (A \otimes B)^T c. \tag{4-8}$$

The unique solution of the normal equation as in (4-8) is $x = (A \otimes B)^+ c$ or equivalently:

$$VecX = (A^{+} \otimes B^{+})VecC = Vec\{B^{+}C(A^{+})^{T}\}. \tag{4-9}$$

Since X and $\{B^{+}C(A^{+})^{T}\}$ are the same order, then $X = B^{+}C(A^{+})^{T}$. $\square$

Now we study some important special cases of (4-1) which are:

Case 1. The Sylvester matrix equation:

$$AX + XB = C, \tag{4-10}$$

where $A \in M_n$, $B \in M_m$ and $C \in M_{n,m}$ are given matrices.

Now, if we use the Vec-notation of (4-10), then by using (1-21) we have the following equivalent equation:

$$(B^{T} \oplus A)VecX = VecC. \tag{4-11}$$

Note that this system has a unique solution if and only if $(B^{T} \oplus A)$ is invertible, that is if and only if none of the eigenvalues of $(B^{T} \oplus A)$ is zero. Now, If $\sigma(A) = \{\lambda_i : i = 1, 2, \ldots, n\}$ and $\sigma(B) = \{\mu_j ; j = 1, 2, \ldots, m\}$, then $\sigma(B^{T} \oplus A) = \{\lambda_i + \mu_j : i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$. So that the Sylvester matrix equation as in (4-10) has a unique solution if and only if A and $(-B)$ have no common eigenvalue if and only if $\sigma(A) \cap \sigma(-B) = \phi$. If on the other hand A and $(-B)$ have an eigenvalue in common, then the existence of the solution depends on the rank of the augmented matrix $[B^{T} \oplus A{:}VecC]$. Note that if $rank[B^{T} \oplus A{:}VecC] = rank\,(B^{T} \oplus A)$, then the solutions exist, otherwise they do not.

Example 4.3. Consider matrices: $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$. Then $\sigma(A) \cap \sigma(-B) = \phi$ and the Sylvester matrix equation as in (4-10) has a unique solution which is $X = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$.

Example 4.4. Consider matrices: $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 5 \\ 2 & -9 \end{bmatrix}$. Then $\sigma(A) \cap \sigma(-B) \neq \phi$, $rank[B^{T} \oplus A{:}VecC] = rank(B^{T} \oplus A) = 3$. Thus the Sylvester matrix equation as in (4-10) has at least one solution which is:

$$X_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix},$$

and any solution can be written a linear combination of X_1 and X_2.

Theorem 4.5. *Let $A \in M_n$, $B \in M_m$ and $C \in M_{n,m}$, and let the all eigenvalues of A and $(-B)$ be negatives. Then the unique solution of the Sylvester matrix equation as in (4-10) is given by:*

$$X = -\int_{0}^{\infty} e^{At}Ce^{Bt}dt. \tag{4-12}$$

Proof. The proof of this Theorem can be found in [14] and we give the proof for the sake of convenience. Since $\sigma(A) \cap \sigma(-B) = \phi$, then the Sylvester matrix equation as in (4-10) has a unique solution. Now by letting $U(t) = 0$ in (3-10) in Theorem 3.2 and integrating both sides of (3-10), we have:

$$X(\infty) - X(0) = A\left[\int_{0}^{\infty} X(t)dt\right] + \left[\int_{0}^{\infty} X(t)dt\right]B. \tag{4-13}$$

Since the all eigenvalues of A and $(-B)$ are negatives, then we have

$$X(\infty) = \lim_{t\to\infty} e^{At}Ce^{Bt} = 0 \text{ and } X(0) = C. \tag{4-14}$$

Now,

$$\begin{aligned} -C &= -[AX + XB] \\ &= A\left[\int_{0}^{\infty} X(t)dt\right] + \left[\int_{0}^{\infty} X(t)dt\right]B. \end{aligned} \tag{4-15}$$

Thus by comparing between both sides of Eq. (4-15), we get the result as in (4-12). $\square$

If we set $B = -A$ and $C = 0$ in the Sylvester equation as in (4-10), then the matrix equation: $AX = XA$ has an infinity solutions and these solutions are the all matrices which commute with A. For example, $X = A^{k}{:}k = 1, 2, \ldots$ are solutions of $AX = XA$.

Case 2. The matrix equation:

$$AX - XA = \alpha X, \tag{4-16}$$

where $A \in M_n$ and $B \in M_m$. This equation has a non-trivial solution if and only if α is an eigenvalue of $(-A^{T} \oplus A)$. Since $\sigma(-A^{T} \oplus A) = \{\lambda_i - \lambda_j : \lambda_i \in \sigma(A)\}$. Hence (4-16) has a non-trivial solution if and only if $\alpha = \lambda_i - \lambda_j$ for some i, j.

Case 3. The matrix equation:

$$AXB = C, \tag{4-17}$$

where $A \in M_{m,n}$, $B \in M_{p,q}$ and $C \in M_{m,q}$ are given matrices. If we use the $Vec(.)$-notation of (4-17) and by using (1-21), then we obtain the following equivalent equation:

$$Hx = c, \tag{4-18}$$

where $H = (B^{T} \otimes A) \in M_{mq,np}$, $x = VecX \in M_{np,1}$, and $c = VecC \in M_{mq,1}$. In particular, if $A \in M_n$, $B \in M_m$ and $C \in M_{n,m}$, then the system $Hx = c$ has a unique solution if and only if $(B^{T} \otimes A)$ is invertible if and only if A and B are invertible. Otherwise we consider a rank argument.

Theorem 4.6. *The matrix equation as in (4-17) has a solution if and only if $rank(A) = m$ and $rank(B) = q$, and the general solution of (4-17) is given by:*

$$X = A^{+}CB^{+} + Q - A^{+}AQBB^{+}, \tag{4-19}$$

for any arbitrary matrix $Q \in M_{n,p}$.

Proof. By using the fact [8] that is if there exists a solution of $Hx = c$, then the general solution is given by $x = H^{+}c + (I - H^{+}H)q$, for any arbitrary vector $q = VecQ \in M_{np,1}$. Now,

$$\begin{aligned} VecX &= (B^{T} \otimes A)^{+}VecC + [I - (B^{T} \otimes A)^{+}(B^{T} \otimes A)]VecQ \\ &= \left\{(B^{+})^{T} \otimes A^{+}\right\}VecC + \left\{I - [(B^{+})^{T} \otimes A^{+}][B^{T} \otimes A]\right\}VecQ \\ &= Vec(A^{+}CB^{+}) + VecQ - Vec(A^{+}AQBB^{+}) \\ &= Vec(A^{+}CB^{+} + Q - A^{+}AQBB^{+}). \end{aligned}$$

Hence, $X = A^{+}CB^{+} + Q - A^{+}AQBB^{+}$, for any arbitrary matrix $Q \in M_{n,p}$. $\square$

Corollary 4.7. *If the matrix equation as in (4-17) has a solution, then the solution is unique* $(X = A^{+}CB^{+})$ *if and only if* $rank(A) = n$ *and* $rank(B) = p$.

Corollary 4.8. *If* $A, X, B, C \in M_n$, $\sigma(A) = \{\lambda_i : i = 1, 2, \ldots, n\}$ *and* $\sigma(B) = \{\mu_j : j = 1, 2, \ldots, m\}$. *Then matrix equation as in (4-17) has a unique solution* $(X = A^{-1}CB^{-1})$ *if and only if* $\lambda_i \neq 0$ *and* $\mu_j \neq 0 : i, j = 1, 2, \ldots, n$.

Theorem 4.9. *If* $rank(A) = n$, $m \geqslant n$ *and* $rank(B) = p$, $q \geqslant p$ *in matrix equation as in (4-17). Then the solution of (4-17) is given by:*

$$X = A^{+}CB^{+}. \tag{4-20}$$

Proof. The least-square problem of $(B^T \otimes A)x = c$: $x = VecX$ and $c = VecC$ is given by:

$$\min_x \|c - (B^T \otimes A)\|_2^2 \equiv \min_X \|C - AXB\|_2^2. \tag{4-21}$$

By applying Theorem 4.2, then we get the solution as in (4-20). $\square$

Example 4.10. Consider matrices:

$$A = \begin{bmatrix} -1 & -1 \\ 0 & 2 \\ 3 & 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $rank(A) = 2 = rank(B)$, then we have

$$A^{+} = (A^T A)^{-1} A^T = \begin{bmatrix} -0.5 & 0 & 0.5 \\ -0.5 & -1 & 0.5 \end{bmatrix}$$

$$B^{+} = (B^T B)^{-1} B^T = \begin{bmatrix} -0.5 & -0.5 \\ 0 & -1 \\ 0.5 & 0.5 \end{bmatrix}.$$

Then the solution of the matrix equation: $AXB = C$ is given by:

$$X = A^{+}CB^{+} = \begin{bmatrix} -0.5 & 0 & 0.5 \\ -0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.5 & -0.5 \\ 0 & -1 \\ 0.5 & 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0.5 \\ 0 & -0.5 \end{bmatrix},$$

and the residual vector is $r = (B^T \otimes A)x - c = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$.

5. Conclusion

The solutions of non-homogeneous matrix differential equations, convolution matrix differential equations and matrix equations which include the renewal matrix equation are presented by using convolution and Kronecker products of matrices. Some important and interesting special cases of these equations are also considered with some illustrated examples. How to extend these methods to find the vector solutions of more general system of non-homogeneous linear matrix and matrix (fractional) differential equations require further research.

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8. Fractional Calculus and Convolution

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